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Broken ergodicity in the self-consistent dynamics of the two-dimensional random sine-Gordon model

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The nonperturbative Hartree approximation is applied to the study of the relaxational dynamics of the two-dimensional random sine-Gordon model. This model describes crystalline surfaces upon disordered substrates, two-dimensional vortex arrays in disordered type II superconducting films, the vortex-free random-field XY model, and other physical systems. We find that the fluctuation-dissipation (FDT) theorem is violated below the critical temperature T_c for large enough times $t > t^*$, where t^* is the "barrier-crossing" time which diverges with the size of the system. Above T_c the dynamics obeys FDT for all times and the local autocorrelation function $q(t)$ diverges as $\sim T \ln t$. The transition is second order for $g < g_{\rm tr}$ where g is the effective coupling to the random-phase periodic potential. In this regime below T_c, as $t \to t^*$, $q(t)$ approaches a finite value $q^*(T)$ [but diverges as $(T_c - T)^{-1}$ as $T \to T_c^-$]. For $g > g_{\rm tr}$ the transition is first order and occurs at the higher g-dependent temperature $T_c(g)$. As $t \to t^*$, the autocorrelations saturate below $T_c(g)$ to a value $q^*(g,T)$ that remains finite as $T \to T_c^-(g)$. In both regimes we find that the ergodic saturation of $q(t)$ to its "one valley" value has the form $q(t) = q^* - ct^{-\nu}$ (as $t \to t^{*-}$). For $t > t^*$ the dynamics is nonergodic. Marginally stable solutions are found within the quasi-FDT approach. They are characterized by a FDT breaking parameter $m(T) = \pi T \left(1 - e^{-4\pi q^*/T}\right) < 1$, [$m(T) = 1$ for $T > T_c$ where FDT holds]. The static correlations behave as $T \ln|\vec{x}|$ for $|\vec{x}| < \xi$ with $\xi \sim \exp[A/(T_c - T)]$. For scales $|\vec{x}| > \xi$ they behave as $(T/m)\ln|\vec{x}|$. Near T_c, $T/m \sim T_c$ but it increases from this value as T is lowered below T_c. The results are compared with dynamic renormalization-group predictions, with equilibrium results obtained by a similar variational approximation with a one-step replica symmetry breaking, and with recent Monte Carlo simulations.

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I. INTRODUCTION

The two-dimensional random-phase sine-Gordon model (RSGM) is currently attracting much attention [1–15]. This intensified activity is due to two main reasons. First, it does describe important physical systems such as vortex-line systems with random pinning [5], the surface of a crystal with a disordered substrate [6], the random-field vortex-free XY model, and others. Second, the advances made in investigations of this model have far-reaching consequences to the understanding of glassy random systems in general. For example, the equilibrium behavior was studied with considerable success by different methods: the renormalization group (RG) [1–4] (both in real and momentum space), Gaussian variational (GV) [7,8] approach which yields replica-symmetry breaking (RSB), $n \to 0$ Bethe ansatz [9,10], and more recently, a replica-free variational approach [11]. All of these approaches agree on the existence of a phase transition into a low-temperature glassy phase but they differ substantially on the very nature of this phase.

The main "fault line" is between the first two approaches where the renormalized perturbative expansion is replica symmetric, contrary to the inherently nonperturbative variational approach. It is not the first time for such a discrepancy to occur in theoretical models. But in the context of random systems it is the best and most studied example. An attempt to reconcile these approaches shows that RSB is a relevant perturbation in the RG calculations [12], along the line of fixed points associated with the glassy phase. The RSB perturbation, however, had to be put in explicitly (and their presence must be traced to nonperturbative effects).

The phase transition was not observed in numerical simulations of the weak-coupling RSGM [13]. Our Monte Carlo (MC) simulations of the surface with disordered substrate [14] did exhibit the transition at the expected temperature. The correlations were found to be in accord with the GV theory. An understanding of a glassy phase requires also the elucidation of the dynamics of the system and its near-equilibrium dynamics in the first place (namely, its relaxation to equilibrium). In view of the equilibrium results [13,14] it is clearly necessary to study the dynamics of the system both by perturbative RG and by a nonperturbative Hartree approximation. The dynamic RG analysis to the lowest nontrivial order was recently completed [6]. The goal of the present work is the detailed investigation of the nonequilibrium dynamics using the Hartree approximation (a brief account of the results in the regime with a continuous transition was given elsewhere [15]). In doing so we follow the footsteps of Horner and collaborators [16–18]. They have shown how the approach of Sompolinsky and Zippelius [19] to spin glasses can be extended to other nonergodic systems which violate the fluctuation-dissipation theorem (FDT). In particular, the concept of quasi-FDT (QFDT) introduced by Horner to study neural networks [16] and applied by Crisanti, Horner, and Sommers to the spherical p-spin interacting spin-glass model [17] was found to be a cornerstone for these investigations. Simply stated it says that the breaking of FDT is the minimal one required to have a (marginally) stable solution for the equations of motion. In this article we apply this principle to the RSGM following its implementation to systems with short-range interactions (e.g., Kinzelbach and Horner for the dynamics of manifolds in disordered media [18]).

Before reviewing the existing results on the RSGM dynamics we shortly revisit the main equilibrium results. The surface terminology will be used throughout this work.

The main quantity of interest is the height-height correlations. Both the RG and the GV agree on the value of T_c, the temperature at which the transition takes place. They also agree on the behavior for $T > T_c$ where the periodic random potential is irrelevant and the behavior is that of a Gaussian ("free") theory with height-height correlations given by

$$C(\vec{x}-\vec{x}') = \left\langle [h(\vec{x}) - h(\vec{x}')]^2 \right\rangle = \frac{T}{\pi} \ln|\vec{x}-\vec{x}'| \qquad (1.1)$$

(assuming the surface stiffness is normalized to one). In these units $T_c = 1/\pi$ and the main disagreement is in the behavior below T_c. Defining $\epsilon = (1 - T/T_c)$, the RG predicts for the correlations

$$C(\vec{x}-\vec{x}') = A(T)\ln|\vec{x}-\vec{x}'| + \frac{\epsilon^2}{\pi^2}\ln^2|\vec{x}-\vec{x}'|. \qquad (1.2)$$

Since the second term dominates at large separations, this behavior has been dubbed (by Toner and DiVincenzo in their study of surfaces in the presence of bulk disorder [4]) as super-rough.

The GV on the other hand predicts a behavior which is logarithmic for $T < T_c$ as well: $S(T)\ln|\vec{x}-\vec{x}'|$ with a different T dependence of the coefficient $S(T)$. Just below T_c this coefficient is predicted to stick to its value at T_c: $S(T_c) = 1/\pi^2$. In one work [8] this behavior is predicted to hold in the whole regime $0 < T < T_c$. Taking into account the finite lattice spacing, another work [7] finds that $S(T)$ increases from this value when T reaches lower temperatures. This increase is more pronounced the larger the coupling is. The replica-free variational approach [11] yields, under certain conditions, $S(T)$ which decreases with T.

For the dynamical behavior the only analytic predictions are from heuristic arguments [20,21] and from dynamic RG calculations [6]. The RG predicts for the high-temperature phase a dynamic exponent $z = 2$ and a finite mobility (defined as the ratio between the average velocity $\langle v\rangle$ and the applied force F in the $F \to 0$ limit). As T approaches T_c from above the mobility vanishes as $|T - T_c|^{1.78}$. Below T_c the dynamics is anomalous: The dynamic exponent increases continuously as $z = 2(1 + 1.78\epsilon)$ (to first order in ϵ). The response becomes nonlinear and $\langle v\rangle \sim F^{z/2}$ for small F. The results of two works [13,22], in which simulations in the presence of a small F were performed, are in qualitative agreement with these predictions. The simulations of the moving surface with substrate disorder by one of us [22] yield a good quantitative agreement as well. The weak-coupling RSGM yields for the anomalous part of the response exponent a linear behavior with ϵ but the coefficient is about one-tenth of the dynamic RG prediction (1.78).

As explained in Sec. II the Hartree approximation begins from the same Martin-Siggia-Rose functional used in the dynamic RG calculations. The self-consistent approach is briefly reviewed and equations of motion for the response and correlation functions are given. The stability of the FDT solutions of these equations is analyzed in Sec. III and the full phase diagram is derived. In Sec. IV the dynamics for $T < T_c$ is investigated first in the early-time ergodic regime and then the late-time nonergodic behavior is explored. From these results we obtain in Sec. V the equal-time equilibrium behavior. In the last section, VI, we summarize the results of this paper and compare them with the ones obtained earlier in the above cited papers.

II. STOCHASTIC DYNAMICS OF THE RSGM: THE HARTREE APPROXIMATION

A. General formalism

We begin with the Hamiltonian of the random sine-Gordon model (RSGM) in $2+1$ dimensions which corresponds to the continuum version of the random discrete Gaussian model in the vicinity of a roughening transition [23]:

$$H = \int d\vec{x} \left\{ \frac{\kappa}{2}\left[\nabla h(\vec{x},t)\right]^2 - g\cos\{\gamma\left[h(\vec{x},t) - d(\vec{x})\right]\} \right\}. \qquad (2.1)$$

The coefficient κ is the stiffness, g is the coupling constant, γ determines the periodicity (we choose $\gamma = 2\pi/a$, a is a lattice spacing), and $d(\vec{x})$ is a random variable such that $\gamma d(\vec{x})$ is a random phase uniformly distributed in the interval $(0, 2\pi]$. Random phases for different $\vec{x}$ are uncorrelated (short-distance correlations will not affect the conclusions). The variable $h(\vec{x},t)$ denotes the height of the surface (or the vortex-line displacement from a periodic lattice).

The dynamics which describes the relaxation of the surface fluctuations in the presence of a stochastic noise is assumed to be governed by the Langevin equation for the field $h(\vec{x},t)$:

$$\frac{1}{\Gamma_0}\frac{\partial h(\vec{x},t)}{\partial t} = -\beta\frac{\delta H[h]}{\delta h(\vec{x},t)} + \zeta(\vec{x},t). \qquad (2.2)$$

The constant Γ_0 is a kinetic coefficient which sets the time scale of the microscopic dynamics and $\beta = 1/T$. The random field $\zeta(\vec{x},t)$ has statistics of a Gaussian white noise with variance

$$\langle\zeta(\vec{x},t)\zeta(\vec{x}',t')\rangle_\zeta = \frac{2}{\Gamma_0}\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{2.3}$$

which ensures the detailed balance, and also that the fluctuation-dissipation theorem is obeyed *locally.* The noise represents the thermal fluctuations in the system.

The next step is to obtain in a standard way the Martin-Siggia-Rose (MSR) [24] generating functional which may directly be averaged over the noise ζ and the quenched disorder $d(\vec{x})$. By the standard techniques, we arrive at the MSR generating functional [$\hat{h}(\vec{x},t)$ is the auxiliary field]:

$$Z_{d,\zeta} = \int \mathcal{D}h\mathcal{D}\hat{h}\exp\Big\{-\int\int d\vec{x}dt(i\hat{h}(\vec{x},t)\left(\partial_t - \Gamma_0\kappa\beta\nabla^2\right)h(\vec{x},t) +\beta\Gamma_0\gamma g i\hat{h}(\vec{x},t)\sin\{\gamma\left[h(\vec{x},t)-d(\vec{x})\right]\})\Big\}\exp\left\{\int\int d\vec{x}dt\;\Gamma_0\zeta(\vec{x},t)i\hat{h}(\vec{x},t)\right\}, \tag{2.4}$$

which still depends on the noise ζ and the randomness d.

The average over noise can be performed using the cumulant expansion:

$$\left\langle\exp\left\{\int d\vec{x}dt\;\Gamma_0\zeta(\vec{x},t)i\hat{h}(\vec{x},t)\right\}\right\rangle_\zeta = \exp\left\{\int d\vec{x}dt\Gamma_0\left[i\hat{h}(\vec{x},t)\right]^2\right\}. \tag{2.5}$$

For simplicity, the Gaussian statistics is also chosen for the random variable $\exp\{i\gamma d(\vec{x})\}$:

$$\langle e^{i\gamma d(\vec{x})}e^{-i\gamma d(\vec{x}')}\rangle_d = a^2\delta(\vec{x}-\vec{x}'). \tag{2.6}$$

Expanding the disorder-dependent factor in Eq. (2.4), averaging all terms, and reexponentiating them again, we obtain $Z = \int \mathcal{D}h\mathcal{D}\hat{h}\exp(-S)$ with [25]

$$S = \int\int d\vec{x}dt\{-\Gamma_0^2[i\hat{h}(\vec{x},t)]^2 + i\hat{h}(\vec{x},t)\left(\partial_t - \Gamma_0\kappa\beta\nabla^2\right)h(\vec{x},t)\} -\frac{1}{4}\beta^2\Gamma_0^2\gamma^2 g^2\int\int\int d\vec{x}dtdt'\;i\hat{h}(\vec{x},t)i\hat{h}(\vec{x},t')\cos\{\gamma\left[h(\vec{x},t)-h(\vec{x},t')\right]\}. \tag{2.7}$$

Note that in Eq. (2.7) we have dropped the contributions from the higher-order harmonics since the essential physical properties are dominated by the first harmonics (higher harmonics are irrelevant in the renormalization-group sense near the transition [23]).

In dynamic calculations, the quantities of interest are correlation functions

$$G(\vec{x}-\vec{x}',t-t') = \langle h(\vec{x},t)h(\vec{x}',t')\rangle, \tag{2.8}$$

and response functions [assuming a time-dependent field acting on $h(\vec{x},t)$]

$$R(\vec{x}-\vec{x}',t-t') = \langle h(\vec{x},t)i\hat{h}(\vec{x}',t')\rangle, \qquad t-t'>0, \tag{2.9}$$

which are calculated from the action (2.7) in the standard way. Causality implies that $R(\vec{x}-\vec{x}',t-t') = 0$ for $t-t' \le 0$. In addition, the correlation function (2.8) is symmetric in time: $G(\vec{x}-\vec{x}',t-t') = G(\vec{x}-\vec{x}',t'-t)$. If we presume that the fluctuation-dissipation theorem holds then G and R are related by

$$\frac{\partial}{\partial t}G(\vec{x},t) = R(\vec{x},t) - R(\vec{x},-t). \tag{2.10}$$

In practice, it is more convenient to work with the Fourier transforms with respect to the spatial variable $\vec{x}$. The Fourier transforms of $G(\vec{x},t)$ and $R(\vec{x},t)$ are defined as

$$G(\vec{x},t) = \int\frac{d\vec{k}}{(2\pi)^2}G(\vec{k},t)e^{-i\vec{k}\cdot\vec{x}}, \tag{2.11}$$

$$R(\vec{x},t) = \int\frac{d\vec{k}}{(2\pi)^2}R(\vec{k},t)e^{-i\vec{k}\cdot\vec{x}}, \tag{2.12}$$

and they will be used extensively in the coming discussions.

B. Equations of motion in the self-consistent approximation

All correlation and response functions can be calculated from the noise and disorder averaged action (2.7). In what follows, we treat S in a self-consistent Hartree-type approximation. This means that all terms which contain more than two fields h or $\hat{h}$ are written in all possible ways as products of pairs of h and $\hat{h}$, and that all such pairs but one are replaced by their expectation values. The action is then by construction quadratic in h and $\hat{h}$ with coefficients which are functions of response and correlation functions only. To carry out this program we apply the field theoretical generalization of the Wick theorem formula:

$$\langle\phi(\vec{x},t)A(h,\hat{h})\rangle = \int d\vec{x}'\,dt'\left\{\langle\phi(\vec{x},t)\hat{h}(\vec{x}',t')\rangle\left\langle\frac{\delta A}{\delta\hat{h}(\vec{x}',t')}\right\rangle + \langle\phi(\vec{x},t)h(\vec{x}',t')\rangle\left\langle\frac{\delta A}{\delta h(\vec{x}',t')}\right\rangle\right\}, \tag{2.13}$$

where $\phi(\vec{x},t) = h(\vec{x},t)$ or $\hat{h}(\vec{x},t)$, and A is any functional of h and $\hat{h}$. Using Eq. (2.13), and after some straightforward algebra, we replace the sine-Gordon term in Eq. (2.7) with

$$\int\int\int d\vec{x}\,dt\,dt'\;i\hat{h}(\vec{x},t)\;i\hat{h}(\vec{x},t')\cos\{\gamma\,[h(\vec{x},t)-h(\vec{x},t')]\}$$

$$\begin{aligned}\to \int\int d\vec{x}\,dt\Big\{& i\hat{h}(\vec{x},t)h(\vec{x},t')\left[-2\gamma^2\int dt'\langle h(\vec{x},t)i\hat{h}(\vec{x},t')\rangle\langle\cos\{\gamma\,[h(\vec{x},t)-h(\vec{x},t')]\}\rangle\right]\\ &+\int dt'\langle i\hat{h}(\vec{x},t)i\hat{h}(\vec{x},t')\langle\cos\{\gamma\,[h(\vec{x},t)-h(\vec{x},t')]\}\rangle\\ &+2\gamma^2 i\hat{h}(\vec{x},t)h(\vec{x},t')\langle h(\vec{x},t)i\hat{h}(\vec{x},t')\rangle\langle\cos\{\gamma\,[h(\vec{x},t)-h(\vec{x},t')]\}\rangle\rangle\Big\}.\end{aligned} \tag{2.14}$$

Introducing the response, $r(\vec{x}-\vec{x}',t-t')$, and correlation, $q(\vec{x}-\vec{x}',t-t')$, functions defined by

$$\begin{aligned}r(\vec{x}-\vec{x}',t-t') &= R(\vec{x}-\vec{x}',t-t')\\ &= \langle h(\vec{x},t)i\hat{h}(\vec{x}',t')\rangle, \qquad t>t',\end{aligned} \tag{2.15}$$

$$\begin{aligned}q(\vec{x}-\vec{x}',t-t') &= G(0,0)-G(\vec{x}-\vec{x}',t-t')\\ &= \langle h(\vec{x},t)h(\vec{x},t)\rangle - \langle h(\vec{x},t)h(\vec{x}',t')\rangle,\end{aligned} \tag{2.16}$$

we arrive at the effective action, which reads

$$\begin{aligned}S_{\text{eff}} = &\int\int d\vec{x}dt\left\{-\Gamma_0^2[i\hat{h}(\vec{x},t)]^2 + i\hat{h}(\vec{x},t)[\partial_t-\Gamma_0\kappa\beta\nabla^2+\mu(0)]h(\vec{x},t)\right\}\\ &-\int\int\int d\vec{x}dtdt'\left\{\frac{1}{2}\omega(t-t')i\hat{h}(\vec{x},t)\;i\hat{h}(\vec{x},t')+\lambda(t-t')i\hat{h}(\vec{x},t)h(\vec{x},t')\right\},\end{aligned} \tag{2.17}$$

where we have defined ($\Gamma_0=1,\gamma=2\pi$)

$$\begin{aligned}\omega(t) &= \frac{1}{2}\beta^2\Gamma_0^2\gamma^2 g^2\langle\cos\{2\pi\,[h(\vec{x},t)-h(\vec{x},t')]\}\rangle\\ &= \frac{2g^2\pi^2}{T^2}e^{-4\pi^2 q(t)},\end{aligned} \tag{2.18}$$

$$\lambda(t) = 4\pi^2\omega(t)r(t). \tag{2.19}$$

Although the original theory described by Eq. (2.7) does not have a mass term, the self-consistent approximation (2.14) generates the term $\mu(0)$ defined by an integral

$$\mu(t) = \int_t^\infty dt'\lambda(t'), \tag{2.20}$$

which might be interpreted as a "mass" term. The contributions generated by this term will significantly affect the dynamic and static properties of the system. The functions μ, ω, and λ depend on t only through $q(t)=q(0,t)$ and $r(t)=r(0,t)$.

Using the standard techniques [18] we now derive the equations of motion for r and q from the action (2.17). In momentum space they read

$$\left[\partial_t+\beta\kappa k^2+\mu(0)\right]r(\vec{k},t) - \int_0^t dt'\lambda(t-t')r(\vec{k},t') = 0, \tag{2.21}$$

with the initial condition

$$r(\vec{k},0^+)=1, \qquad r(\vec{k},t)=0 \quad \text{for } t\le 0, \tag{2.22}$$

and

$$[\partial_t+\beta\kappa k^2+\mu(0)]q(\vec{k},t) - \int_0^t dt'\lambda(t-t')q(\vec{k},t') = 1-[I(\vec{k},t)-I(\vec{k},0)], \tag{2.23}$$

with

$$q(\vec{k},0)=0. \tag{2.24}$$

$I(\vec{k},t)$ is a shorthand notation for the integral

$$I(\vec{k},t) = \int_0^\infty dt'[\omega(t+t')r(\vec{k},t')-\lambda(t+t')q(\vec{k},t')]. \tag{2.25}$$

III. STABILITY ANALYSIS AND PHASE DIAGRAM

The coupled equations of motion (2.21) and (2.23) derived in the self-consistent Hartree-type approximation uniquely determine the time evolution of the correlation and response functions of the system. The additional relation between q and r comes from the FDT (2.10) which now reads

$$\partial_t q(\vec{k},t) = r(\vec{k},t), \tag{3.1}$$

and which implies the corresponding relation between ω and λ:

$$\partial_t \omega(t) = -\lambda(t). \tag{3.2}$$

If FDT holds, $I(\vec{k},t) - I(\vec{k},0) = 0$ and equations for r and q become equivalent, as they should.

For the dynamics at high temperatures, the FDT given by Eq. (3.1) is expected to hold for all times. At low temperatures, however, the ergodicity might be broken with important consequences on the behavior of the correlation and response functions. The nonergodicity is related to the possibility of the existence of a phase with glassy properties and it would be manifested only at large times compared to some time scale t^* determined by the system size. Therefore we first investigate the solutions of Eqs. (2.21) and (2.23) in the limit $t \to \infty$ assuming that the FDT in the form (3.1) holds for all times.

It is convenient to start from Eq. (2.23), which together with Eq. (3.1) yields

$$r(\vec{k},t) + \left[\beta\kappa k^2 + \mu(t)\right] q(\vec{k},t) + \int_0^t dt' \, [\omega(t-t') - \omega(t)] r(\vec{k},t') = 1. \tag{3.3}$$

$r(\vec{k},t)$ is positive and vanishes as $t \to \infty$. In addition it implies that $\partial_t \omega(t) \le 0$. The contribution of the integral is positive and negligible as $t \to \infty$. Hence the following inequality holds:

$$[\beta\kappa k^2 + \mu(t)] q(\vec{k},t) \le 1. \tag{3.4}$$

After integration over the first Brillouin zone in $\vec{k}$ space, $|k_x|, |k_y| \in [\pi/L, \pi/a]$, the condition (3.4) may be written as

$$q(t) \le \frac{T}{4\pi} \ln\left[\frac{\pi^2 + \mu(t)/\beta}{(\pi/L)^2 + \mu(t)/\beta}\right], \tag{3.5}$$

where without loss of generality we set $a = 1$ and $\kappa = 1$ (their values will be restored).

Note that from the FDT relation and Eq. (2.20) it also follows that

$$\mu(t) = \omega(t) - \omega_0, \tag{3.6}$$

where $\omega_0 = \omega(t = \infty)$ is defined by Eq. (2.18) with the stationary value of $q(t)$, $q(t) \to q_0 = (T/4\pi)\ln(L^2)$ as $t \to \infty$.

Introducing a new function $\Delta_L(q,T)$ defined by

$$\Delta_L(q,T) = \frac{\pi^2}{T}\left[\frac{1-(1/L^2)e^{4\pi q/T}}{e^{4\pi q/T}-1} - \frac{2g^2}{T}e^{-4\pi^2 q}\right] + \frac{2\pi^2 g^2}{T^2}\left(\frac{1}{L^2}\right)^{\pi T}, \tag{3.7}$$

the condition (3.5) reads

$$\Delta_L(q,t) \ge 0 \qquad \text{for all } q. \tag{3.8}$$

Equation (3.8) is the stability condition derived from the equations of motion under the assumption that the FDT holds. Therefore all solutions of Eqs. (2.21) and (2.23) for q and r, related by Eq. (3.1), should satisfy Eq. (3.8). The failure of (3.8) would mean that FDT solutions are not applicable for the description of a dynamically stable system.

For simplicity, we will analyze the condition (3.8) in the limit $L \to \infty$. It turns out that the analysis is particularly simple if we introduce a new variable z,

$$z \equiv e^{-4\pi q/T} \in [0,1], \tag{3.9}$$

in terms of which Eq. (3.8) becomes

$$\Delta(q,T) \equiv \Delta_{L\to\infty}(q,T) = \frac{2\pi^2 g^2}{T^2}\frac{z}{1-z}\left[\frac{T}{2g^2} - z^{\pi T-1} + z^{\pi T}\right] \ge 0. \tag{3.10}$$

Since q diverges in the $t \to \infty$ limit, this inequality should be studied in the $z \to 0$ limit. The prefactor $z/(1-z)$ is always positive. To understand the behavior of the terms in brackets, we consider the cases $\pi T < 1$ and $\pi T > 1$ separately.

If $\pi T < 1$, the dominant term $z^{\pi T-1}$ diverges at $z \to 0$. Since this term is negative, the condition $\Delta \ge 0$ is always broken (or the theory is unstable) independently of g.

If $\pi T > 1$, all terms in Eq. (3.10) are important. Since $z \in [0,1]$, $1 \ge z^{\pi T-1} \ge z^{\pi T}$ and Δ may change its sign if $T/(2g^2) < 1$. If $T/(2g^2) \ge 1$, Δ is always positive, which means that the theory is stable. There is the regime of g, therefore, where FDT solutions become unstable even for $T > 1/\pi$. From $2g^2 > T > 1/\pi$ it follows that a minimal g for which this might happen satisfies

$$g \ge \frac{1}{\sqrt{2\pi}}. \tag{3.11}$$

To summarize, a simple analysis shows that for $T < 1/\pi$ the condition (3.10) is always violated regardless of g, while for $T > 1/\pi$ the condition may be either violated or fulfilled depending on the value of g. To proceed further, it is illustrative to plot the function $\Delta_L(q,T)$ for g which is larger and smaller than the characteristic value $1/\sqrt{2\pi}$.

Figure 1 shows the behavior of $\Delta_L(q,T)$ for several temperatures in the vicinity of the critical temperature $T_c = 1/\pi$, and $g = 0.3 \approx (3/4)(1/\sqrt{2\pi})$, $L = 10^3$. Increasing the system size L, the general nature of the curves is not affected. For the largest possible q, $q = q_0$, $\Delta_L(q,T)$ vanishes for all T. Decreasing T below $1/\pi$, $\Delta_L(q,T)$ becomes negative (unstable) at some values of q, which continuously approaches q_0 as $T \to 1/\pi$ from

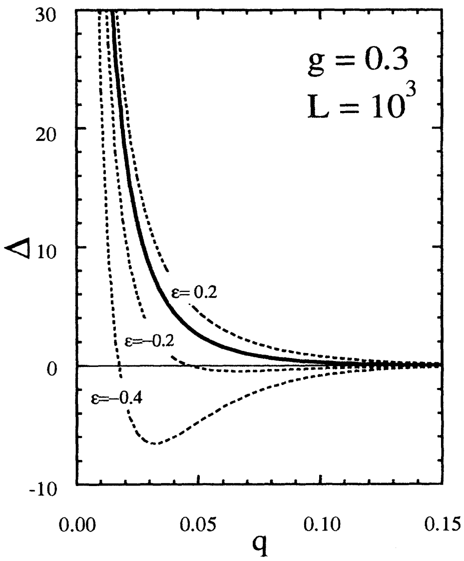

FIG. 1. Plot of $\Delta_L(q,T)$ versus q for different temperatures $T = T_c(1+\epsilon)$, and $g < g_{\rm tr}$. For T larger than T_c (or for $\epsilon > 0$), $\Delta_L(q,T)$ is always positive and for large $q \to q_0$, it approaches zero from above. Full line is $\Delta_L(q,T)$ for $\epsilon = 0$. Below T_c, $\Delta_L(q,T)$ is negative, indicating the instability of the FDT solutions.

below.

In Fig. 2 we show curves for $g = 0.5 \approx (5/4)(1/\sqrt{2\pi})$. Now, decreasing the temperature, $\Delta_L(q,T)$ begins to be negative at some finite q which is discontinuously shifted from q_0 and it stays finite in the limit $L \to \infty$.

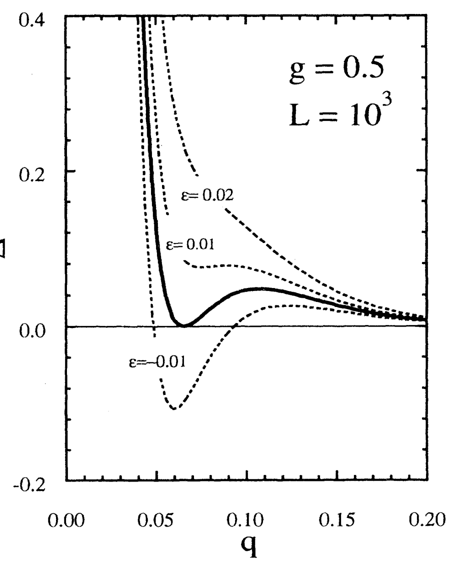

FIG. 2. Typical behavior of $\Delta_L(q,t)$ as a function of q for $T = T_c(1+\epsilon)$ and $g > g_{\rm tr}$. The minimum of $\Delta_L(q,T)$ is the place where it first becomes negative for decreasing T below T_c.

The critical values q_c and T_c for which the minimum of $\Delta_L(q,T)$ first becomes negative are obviously defined by the equations

$$\Delta(q_c,T_c) = 0, \tag{3.12}$$

$$\Delta'(q_c,T_c) = 0. \tag{3.13}$$

For $g > 1/\sqrt{2\pi}$, the above system of equations has two solutions: the trivial solution $q_c = q_0$ which diverges in the limit $L \to \infty$, and $q_c < q_0$ which in the $L \to \infty$ limit approaches the value

$$q_c = \frac{T}{4\pi} \ln\left(\frac{\pi T_c}{\pi T_c - 1}\right), \tag{3.14}$$

where T_c and g are related by the equation

$$g^2 = \frac{1}{2\pi} \frac{(\pi T_c)^{\pi T_c + 1}}{(\pi T_c - 1)^{\pi T_c - 1}}. \tag{3.15}$$

For $g < 1/\sqrt{2\pi}$, Eqs. (3.12) and (3.13) have only a single solution: $q_c = q_0$.

The plots of $\Delta_L(q,T)$ presented in Fig. 1 and Fig. 2 are characteristic plots for any $g < 1/\sqrt{2\pi}$ and $g > 1/\sqrt{2\pi}$, respectively. The fundamental difference between these two situations is that in the former case the minimum of Δ_L moves continuously toward q_0 as $T \to T_c$, while in the latter case it saturates at some value of q which is at $T = T_c$ given by Eq. (3.14). We will see shortly that for $g < 1/\sqrt{2\pi}$ the system has a second-order transition while for $g > 1/\sqrt{2\pi}$ it becomes a first-order one. Hence the value

$$g_{\rm tr} = \frac{1}{\sqrt{2\pi}} \tag{3.16}$$

is termed the tricritical value of g. Figure 3 shows the

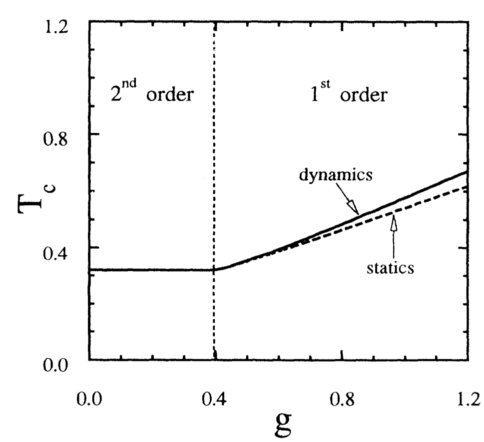

FIG. 3. Phase diagram of the RSGM. Vertical dotted line $g = g_{\rm tr}$ separates regimes of the first- and second-order transition. The full line is the dynamic result given by formula (3.15). For comparison, the static result of Korshunov in Ref. [7] is plotted by a broken line.

phase diagram, i.e., the dependence of T_c on g. In the regime of second-order transition ($g < g_{\rm tr}$), the critical temperature is always $T_c = 1/\pi$ while in the case of a first-order transition ($g > g_{\rm tr}$) T_c increases as g increases according to Eq. (3.15).

The predicted phase diagram is in agreement with one calculated in Ref. [7] on the basis of the Gaussian variational method with a one-step symmetry breaking mechanism. Both calculations result in the same tricritical value $g_{\rm tr}$ and $T_c - g$ dependence for $g \le g_{\rm tr}$. For larger g, the static T_c becomes slightly lower than the dynamic one. This difference increases with the increase of g. The difference between the two T_c is due to the fact that the FDT broken solution has a higher equilibrium free energy than the FDT one.

To gain a better understanding of the system dynamics we can, at least approximately, try to numerically integrate Eq. (2.23) assuming that the FDT holds everywhere. We start from the form of the free propagator of the action (2.17) which is expected to be a good solution in high-temperature phase (where the cosine term is irrelevant). This is the same approximation which leads (2.23) to (3.4). It yields

$$q(\vec{k},t) = \frac{1}{\beta k^2 + \mu(t)}\left(1 - e^{-[\beta k^2 + \mu(t)]t}\right). \quad (3.17)$$

The effect of the lattice and the disorder is partially included through the mass term $\mu(t)$ which does not exist in a free theory. Equation (3.17) can be analytically integrated over $\vec{k}$ space, and an integration yields a self-consistent equation for the correlation function $q(t)$ [or mass $\mu(t)$, Eq. (3.6)]:

$$q(t) = \frac{T}{4\pi}\left\{\ln\left(\frac{\pi^2 + \mu(t)/\beta}{(\pi/L)^2 + \mu(t)/\beta}\right) - E_1\{[(\pi/L)^2 + \mu(t)/\beta]\beta t\} + E_1\{[\pi^2 + \mu(t)/\beta]\beta t\}\right\}, \quad (3.18)$$

where $E_1(x)$ is the exponential integral

$$E_1(x) = \int_x^\infty dx' \, \frac{e^{-x'}}{x'}. \quad (3.19)$$

The numerical solutions of Eq. (3.18) for temperatures close to T_c are shown in Fig. 4 (for $g = 0.3 < g_{\rm tr}$) and in Fig. 5 (for $g = 0.5 > g_{\rm tr}$). For $T > T_c$ and long times t, $q(t)$ has logarithmic behavior $q(t) \sim \ln(t)$ up to some t where the evolution of $q(t)$ is affected by the system size, $q \to q_0 \sim \ln(L)$.

As we already noticed, the behavior of q below T_c is quite different for $g < g_{\rm tr}$ and $g > g_{\rm tr}$. In the figures we see that for $g < g_{\rm tr}$, q as a function of time continuously approaches its asymptotic value as $T \to T_c$. For $g > g_{\rm tr}$, on the other hand, there is a finite jump between q_c and q_0. In this sense the transition is of the second order (continuous) or of the first order (discontinuous).

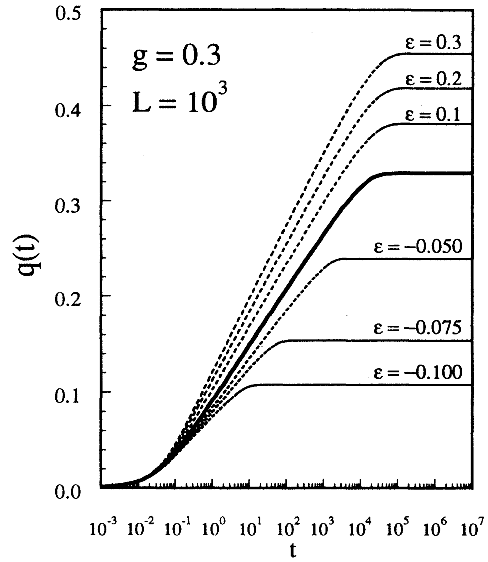

FIG. 4. Typical behavior of the solutions of Eq. (3.18) for $g < g_{\rm tr}$ and $T = T_c(1+\epsilon)$, where $T_c = 1/\pi$. For $T > T_c$, the large-t saturation of the curves $q(t)$ is a consequence of finite size effects while for $T < T_c$ it is also affected by nonergodicity. The full line is $q(t)$ at T_c.

IV. THE DYNAMICS IN THE GLASSY PHASE

As we have seen in the preceding section, the FDT dynamics has to be violated at low temperatures $T < T_c$. In the present section we analyze the dynamics in this phase. The general approach we take is following that of

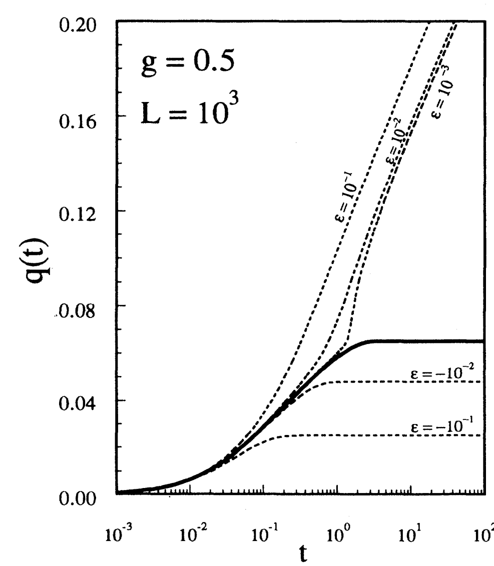

FIG. 5. Plot of the solutions of Eq. (3.18) for $T = T_c(1+\epsilon)$. For $g = 0.5$ critical values are $T_c = 0.3521$ and $q_c = 0.0656$.

Sompolinsky and Zippelius [19]. We assume the size of the system to be large and finite and the limit $t \to \infty$ is taken before the thermodynamic one. Hence the free-energy barriers are also large but finite and the time scale t^* for the system to go over the barrier separates between two dynamic regimes: For $t < t^*$ the relaxation occurs within each phase-space component ("valley") separately and the dynamics is ergodic. For $t > t^*$ the system goes over the barriers from one component to another. In this regime the dynamics violates the FDT. The assumption is that $t^* \to \infty$ in the thermodynamic limit but in such a way that larger time scales $t > t^*$ are still defined such that $t/t^* \to \infty$ as well in this limit. Although the dependence of t^* on L is not known, the value of $q(t^*) = q^*$ can be found and since $q(t)$ is monotonically increasing the dynamics can be analyzed in terms of $q < q^*$ (ergodic) and $q > q^*$ (nonergodic regime). This will be achieved through the QFDT hypothesis. It is also important to note from the outset that the dynamics in the nonergodic regime feeds back on the short-time dynamics because the FDT violation leads to the generation of an anomalous "mass" term which is present at all times.

Indeed, once the FDT is broken, $\lambda(t) \neq -\partial\omega/\partial t$ and therefore $\int_{t^*}^{\infty} \lambda(t)dt$ is larger than $\omega(\infty) - \omega(t^*)$ and it will be assumed that $\mu^* = \lim_{t^*\to\infty} \int_{t^*}^{\infty} \lambda(t)dt > 0$ due to the contribution from the non-FDT regime. Looking back at the instability found in Eq. (3.8) it can be seen that the presence of $\mu^* > 0$ may prevent the instability since a finite μ^* will change $\Delta(q)$ to $\tilde{\Delta}(q) = \Delta(q) + \mu^*$. So the stability analysis has to be performed on $\tilde{\Delta}(q)$. If we denote its minimum by $q_{\min}$, the stability condition will be $\tilde{\Delta}(q_{\min}) \geq 0$. The QFDT hypothesis states that $\tilde{\Delta}(q_{\min}) = 0$, which means that $\mu^* = -\Delta(q_{\min})$ is the minimal necessary to marginally stabilize the behavior. This condition also leads very naturally to the identification of q^* with $q_{\min}$ and therefore q^* is the solution of the equation

$$\Delta'(q^*) = 0. \tag{4.1}$$

We now discuss the dynamics in each regime separately.

A. Ergodic dynamics $t < t^*$

The numerical solutions of Eq. (4.1) for temperatures $T \leq T_c$ and for different values of constant g are shown in Fig. 6. In the regime of continuous transitions, as T approaches $T_c = 1/\pi$ from below, q^* diverges as

$$q^* = \frac{T}{4\pi} \frac{1}{1 - \pi T} \ln\left(\frac{1}{2\pi g^2}\right). \tag{4.2}$$

For discontinuous transitions, the dependence of T_c on g is given by Eq. (3.15). As $T \to T_c$, q^* approaches q_c given by Eq. (3.14). The final points $T = T_c$ of the curves for given $g > g_{\rm tr}$ shown in Fig. 6 indeed end up on the curve given by Eq. (3.14).

Since on finite time scales $t < t^*$ the FDT solutions are still stable, the evolution of $q(t)$ toward q^* is described

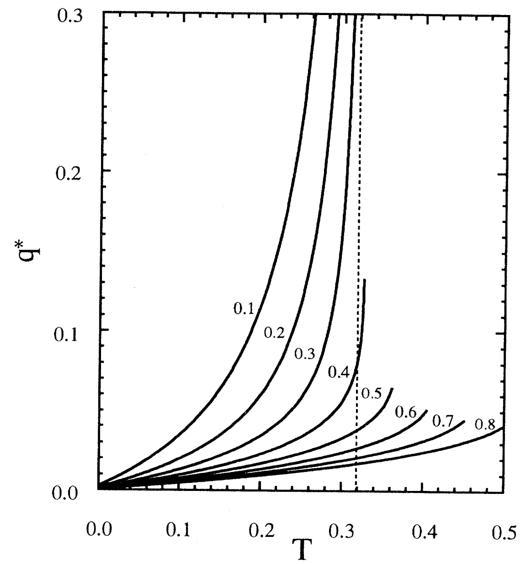

FIG. 6. Plot of q^* versus T for $T \leq T_c$. At $T = T_c$ the curves diverge for $g = 0.1$, 0.2, $0.3 < g_{\rm tr}$, while for $g = 0.4$, 0.5, 0.6, 0.7, $0.8 > g_{\rm tr}$ they have finite values.

by Eq. (2.23). Expanding Eq. (2.23) around q^* we can investigate how $q(t)$ approaches its stationary value q^*:

$$q(\vec{k}, t) = q^*(\vec{k}) + \delta q(\vec{k}, t). \tag{4.3}$$

An expansion (4.3) is inserted into the equation of motion (for details see the appendix of Ref. [18]), which after lengthy manipulations yields

$$\left\{1 - \pi T\left(1 - e^{-4\pi q^*/T}\right)\right\} \delta q^2(t) + \int_0^t dt'[\delta q(t-t') - \delta q(t)]\partial_{t'}\delta q(t') = 0. \tag{4.4}$$

The asymptotic solution of the above equation is assumed to have an algebraic form

$$\delta q(t) \sim t^{-\nu}. \tag{4.5}$$

The exponent ν is a solution of the equation obtained by inserting (4.5) into Eq. (4.4):

$$\frac{\Gamma^2(1-\nu)}{\Gamma(1-2\nu)} = \pi T\left(1 - e^{-4\pi\kappa q^*/T}\right), \tag{4.6}$$

where Γ denotes the gamma function. The right-hand side will turn out to be equal to the FDT breaking parameter $m(T)$ (see below).

The plot of the temperature-dependent exponent ν for some values of g is given in Fig. 7. In the weak-coupling regime (where the transition is continuous) the right-hand side of Eq. (4.6) is very close to $\pi T = T/T_c$ because the second term in parentheses is exponentially small and

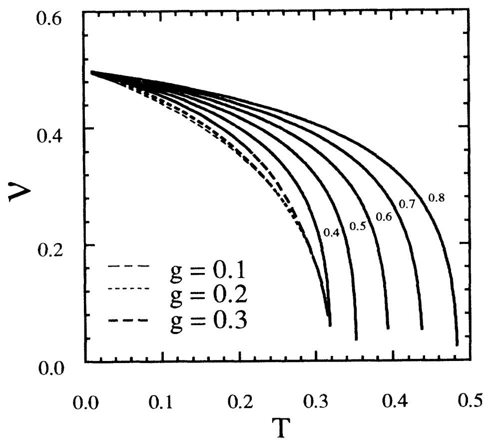

FIG. 7. The temperature dependence of the critical exponent ν for different coupling constants g.

hence ν becomes g independent.

The crossover time characterizing how $q(t)$ approaches q^* diverges as $\epsilon \to 0^+$ (or $T \to T_c^-$) and most of the time $t < t^*$ is spent by the system in the "critical" regime with anomalously slow dynamics. Actually the dynamics becomes even slower since it changes from a power to a logarithmic law. To see that we note that the right-hand side of Eq. (4.6) approaches 1 as $\epsilon \to 0^+$. In the weak-coupling regime it is $1-\epsilon$ [the results will also hold at the first-order regime with a more cumbersome dependence of the right-hand side on $(T_c - T)$].

Looking at the equation $\Gamma^2(1-\nu)/\Gamma(1-2\nu) = 1-\epsilon$ we find

$$\nu = \frac{\sqrt{6}}{\pi}\epsilon^{1/2}. \tag{4.7}$$

Hence

$$\delta q(t) \sim t^{-\frac{\sqrt{6}}{\pi}\epsilon^{1/2}} \tag{4.8}$$

and if $\epsilon \to 0$ such that $|\epsilon^{1/2}\ln t| \ll 1$ this behavior implies

$$\delta q(t) \sim c' \ln t \tag{4.9}$$

with an ϵ-dependent coefficient c'.

It is interesting to note that the logarithmic behavior can also be obtained as $T \to T_c^+$. For simplicity, we analyze the $g > g_{\rm tr}$ case (note that for $g < g_{\rm tr}$ L must be kept finite but for $g > g_{\rm tr}$ this limitation is not necessary). Assuming δq and $\delta\Delta = \Delta(q) - \Delta(q^*)$ to be small, a solution of the form (4.3) where $\delta q \sim |\delta\Delta|^{1/2} f(t/t_c)$ is looked for. Similarly as in the derivation of Eq. (4.4), the expansion of (2.23) yields an equation for $f(u)$, $u = t/t_c$:

$$\int_0^u du'\, [f(u-u') - f(u)] f'(u') = -1, \tag{4.10}$$

of which the solution is

$$f(u) = \frac{\sqrt{6}}{\pi}\ln(u). \tag{4.11}$$

The time scale t_c has to be determined by matching the solution (4.12) with the solution at short time scale, $t \sim 1$, which gives the estimate

$$t_c \sim e^{1/|\epsilon|^{1/2}}. \tag{4.12}$$

B. Nonergodic dynamics $t > t^*$

In this regime FDT is violated and the relation between $r(t)$ and $q(t)$ changes. The simplest change that can be implemented is to assume

$$r(\vec{k},t) = m\frac{\partial}{\partial_t}q(\vec{k},t), \qquad t \gg t^*. \tag{4.13}$$

One can take this relation as an alternative definition of QFDT. This will imply $r(t) = m\partial_t q(t)$, and

$$\lambda(t) = -m\frac{\partial}{\partial_t}\omega(t), \qquad t \gg t^*. \tag{4.14}$$

Since the generation of the mass μ^* [see Eq. (2.14)] requires $|\lambda(t)| < |\partial\omega/\partial t|$ the QFDT solution is possible only if $m < 1$. The parameter m is a free parameter which has to be determined self-consistently from the dynamics for times below and above t^*. In general, it can depend on the temperature or other properties of the studied system (for example, g). For $m \to 1$, the QFDT reduces to the ordinary FDT.

For the purpose of future discussion, it is convenient to cast the equations of motion (2.21) and (2.23) in such a form that contributions from the $t < t^*$ and $t > t^*$ regimes can be easily identified. After some algebra, Eq. (2.21) for the response function reads

$$\left\{\partial_t + \beta k^2 + \mu^* + \int_0^{t^*} dt'\lambda(t')\right\} r(\vec{k},t) - \int_{t^*}^{t} dt'\lambda(t-t')r(\vec{k},t') = \lambda(t)q^*(\vec{k}). \tag{4.15}$$

Using the QFDT (4.13), the integrals $I(\vec{k},t)$ and $I(\vec{k},0)$ in Eq. (2.23) become

$$I(\vec{k},0) = (1-m)\omega^* q^*(\vec{k}) + m\omega_0 q_0, \tag{4.16}$$

$$I(\vec{k},t) = (1-m)\omega(t)q^*(\vec{k}) + m\omega_0 q_0(\vec{k}) - \lambda(t)\int_0^{t^*} dt' q(\vec{k},t'), \tag{4.17}$$

and the equation for the correlation function reads

$$\left\{\partial_t + \beta k^2 + \mu^* + \int_0^{t^*} dt'\lambda(t')\right\} q(\vec{k},t) - \int_{t^*}^{t} dt'\lambda(t-t')q(\vec{k},t')$$

$$= 1 - (1-m)[\omega(t) - \omega^*]q^*(\vec{k}) + 2\lambda(t)\int_0^{t^*} dt' q(\vec{k},t'), \quad (4.18)$$

where q_0 is the asymptotic value $q(t \to \infty)$. Note that for $T < T_c$ it is different from the one mentioned before in the $T > T_c$ case. Equations (4.15) and (4.18) are equivalent to one another, as they have to be because of the QFDT relation between them.

To proceed further, we rescale the variable $t \to u = t/t_0$ where $t_0 \gg t^*$ is the time needed to reach the equilibrium distribution and it diverges as $L \to \infty$, and look for the solutions for the response and correlation functions which have the scaling forms

$$q(\vec{k},t) = \tilde{q}(\vec{k},t/t_0), \quad (4.19)$$

$$r(\vec{k},t) = \frac{1}{t_0}\tilde{r}(\vec{k},t/t_0), \quad (4.20)$$

which imply that

$$\omega(t) = \tilde{\omega}(t/t_0), \quad (4.21)$$

$$\lambda(t) = \frac{1}{t_0}\tilde{\lambda}(t/t_0). \quad (4.22)$$

Starting from Eqs. (4.15) and (4.18), a straightforward calculation gives the equations for $\tilde{q}(\vec{k},t/t_0)$ and $\tilde{r}(\vec{k},t/t_0)$ for times $t \gg t^*$:

$$\left(\beta k^2 + \mu^*\right)\tilde{r}(\vec{k},u) - \int_0^u du'\ \tilde{\lambda}(u-u')\tilde{r}(\vec{k},u')$$

$$= \tilde{\lambda}(t)q^*(\vec{k}) \quad (4.23)$$

and

$$\left(\beta k^2 + \mu^*\right)\tilde{q}(\vec{k},u) - \int_0^u du'\ \tilde{\lambda}(u-u')\tilde{q}(\vec{k},u')$$

$$= 1 - (1-m)\left[\tilde{\omega}(u) - \omega^*\right]q^*(\vec{k}). \quad (4.24)$$

Again, Eqs. (4.23) and (4.24) are equivalent because of the QFDT.

We do not intend to look for the solutions of Eqs. (4.23) and (4.24) for arbitrary u. We are primarily interested in their asymptotic behavior for $u \to 1$ (or $t \to t_0 \to \infty$). Because of the self-consistency, however, the solutions of the above equations in the limit $u \to 0$ have to be matched with the solutions on finite time scales, when $t \to t^*$ from below. The matching condition will give an equation for μ^* (or q^*).

For $u \to 0$, the contribution from the integral in Eq. (4.23) vanishes and the equation yields

$$\left(\beta k^2 + \mu^*\right)\tilde{r}(\vec{k},0) = \tilde{\lambda}(0)q^*(\vec{k}). \quad (4.25)$$

If we now substitute the expression $q^*(\vec{k}) = 1/(\beta k^2 + \mu^*)$, and assume the existence of a nontrivial solution, $\tilde{r}(\vec{k},0) \neq 0$, Eq. (4.25) gives the condition on μ^* which is actually the defining equation for μ^*:

$$1 = 4\pi^2\omega^* \int \frac{d\vec{k}}{(2\pi)^2}\frac{1}{\left(\beta k^2 + \mu^*\right)^2}. \quad (4.26)$$

This is exactly the condition $\Delta'(q^*) = 0$ which was heuristically argued before, Eq. (4.1). To show it, we have to integrate out Eq. (4.24) (keeping L finite) and use the definition of $\Delta(q,T)$, i.e., Eq. (3.7). In the replica language it is the condition on μ^* for the eigenvalue of the replicon mode to change from a negative value to zero [see Eq. (5) in Ref. [8]].

The matching condition for the solutions on two time scales also fixes the parameter m. Equation (2.20) and QFDT (4.14) give

$$\mu^* = \int_{\mu^*}^{\infty} dt'\lambda(t') = m\omega^*, \quad (4.27)$$

or $m = \mu^*/\omega^*$. Using the condition $\Delta'(q^*,T) = 0$, we obtain

$$m(T) = \pi T\left(1 - e^{-4\pi q^*/T}\right), \quad (4.28)$$

which appeared on the right-hand side of Eq. (4.6).

The dependence of m on the coupling constant is hidden in q^*. For $g < g_{\rm tr}$ and at $T = T_c = 1/\pi$, the exponential term in (4.28) has nonanalytic behavior. The asymptotic form of m as $T \to T_c$ is given by

$$m(T) = \pi T\left[1 - \left(2\pi g^2\right)^{T_c/(T_c - T)}\right]. \quad (4.29)$$

In order to find the asymptotic behavior of $q(t)$ in the $t \to t_0 \to \infty$ limit, it is convenient to cast Eq. (4.24) in the form

$$\left\{\beta k^2 + \mu(t) + \int_0^t dt'\lambda(t')\right\}q(\vec{k},t) - q(\vec{k},t)\int_0^t dt'\lambda(t')$$

$$= 1 - (1-m)\left[\omega(t) - \omega^*\right]q^*(\vec{k}), \quad (4.30)$$

where we have used the fact that $t_0 \gg t^*$. The $t_0 \to \infty$

limit yields

$$\left(\beta k^2 + \mu_0\right) q_0(\vec{k}) = 1 - (1-m)(\omega_0 - \omega^*) q^*(\vec{k}). \quad (4.31)$$

Equation (4.31) will be used in the next section in the derivation of the static expression for the height-height correlation functions.

V. STATICS: HEIGHT-HEIGHT CORRELATIONS

The scaling properties of the rough surface are manifested in the behavior of the height-height correlation function which is defined by

$$C(\vec{x} - \vec{x}', t - t') = \left\langle [h(\vec{x}, t) - h(\vec{x}', t')]^2 \right\rangle, \quad (5.1)$$

where, as in Eqs. (2.8) and (2.9), the brackets $\langle\rangle$ denote average over stochastic dynamics and over the quenched disorder. In terms of the previously introduced correlation function $G(\vec{x} - \vec{x}', t - t')$, $C(\vec{x} - \vec{x}', t - t')$ reads

$$C(\vec{x} - \vec{x}', t - t') = 2\left[G(0,0) - G(\vec{x} - \vec{x}', t - t')\right]. \quad (5.2)$$

Of particular interest is the behavior of $C(\vec{x} - \vec{x}', t - t')$ in the static limit, i.e., for $t = t'$, which is the most commonly measured quantity in either experimental or numerical studies. In this limit, expression (5.1) reduces to

$$C(\vec{x} - \vec{x}) = 2 \int \frac{d\vec{k}}{(2\pi)^2} \left\{1 - e^{-i\vec{k}\cdot(\vec{x} - \vec{x}')}\right\} G(\vec{k}, 0). \quad (5.3)$$

The equation for $G(\vec{k}, 0)$ can be derived from the effective action (2.17) by standard techniques [18] similar to that for Eq. (2.23) for $q(\vec{k}, t)$. Actually, what we need in Eq. (5.3) is only $G(\vec{k}, t)$ at $t = 0$ which is given by the initial condition

$$G(\vec{k}, 0) = \frac{1}{\beta k^2}\left[1 + I(\vec{k}, 0)\right]. \quad (5.4)$$

To calculate the integral $I(\vec{k}, 0)$, we need to know the dynamics on all time scales. Since the dynamics are fundamentally different above and below T_c, we treat these two cases separately.

(a) $T > T_c$: In this regime the FDT holds for all times. Straightforward algebra yields

$$I(\vec{k}, 0) = \frac{\omega_0}{\beta k^2}. \quad (5.5)$$

In the thermodynamic limit, (5.5) vanishes because $\omega_0 \to 0$ if $L \to \infty$. With Eqs. (5.4) and (5.5), the $C(\vec{x} - \vec{x}')$ may be written as

$$C(\vec{x} - \vec{x}) = \frac{2}{\beta} \int \frac{d\vec{k}}{(2\pi)^2} \frac{1 - \cos[\vec{k}\cdot(\vec{x} - \vec{x}')]}{k^2}. \quad (5.6)$$

For large $|\vec{x} - \vec{x}'|/a$ the dominant contribution in the integral comes from the region $k|\vec{x} - \vec{x}'| \gg 1$, which yields the asymptotic behavior

$$C(\vec{x} - \vec{x}) = \frac{T}{\pi} \ln\left(\frac{|\vec{x} - \vec{x}')}{a}\right), \qquad |\vec{x} - \vec{x}'| \gg a. \quad (5.7)$$

The logarithmic behavior of $C(\vec{x} - \vec{x}')$ for large $|\vec{x} - \vec{x}'|$ with a prefactor proportional to temperature is the well-known property of the high-temperature phase.

(b) $T < T_c$: This is the regime of nonergodic dynamics. The QFDT has to be applied here. In this case, the integral $I(\vec{k}, 0)$ is given by Eq. (4.16), and the expression for $G(\vec{k}, 0)$ reads

$$G(\vec{k}, 0) = \frac{1}{\beta k^2}\left[1 + (1-m)\omega^* q^* + m\omega_0 q_0\right]. \quad (5.8)$$

Using

$$\mu_0 - \mu^* = \int_{t_0}^{t^*} dt'\ \lambda(t') = m(\omega_0 - \omega^*), \quad (5.9)$$

and Eq. (4.31), we can derive a useful relation:

$$q_0 = \frac{1}{m} \frac{1}{\beta k^2 + \mu_0} + \left(1 - \frac{1}{m}\right) q^*, \quad (5.10)$$

with the help of which $G(\vec{k}, 0)$ can be written in the form

$$G(\vec{k}, 0) = \frac{1}{\beta k^2}\left(\frac{1}{m} + \frac{\omega_0}{\beta k^2 + \mu_0}\right) + \left(1 - \frac{1}{m}\right)\frac{1}{\beta k^2 + \mu^*}. \quad (5.11)$$

Again, in the thermodynamic limit ω_0 vanishes. The asymptotic form of the height-height correlation function can be obtained by inserting Eq. (5.11) into (5.3) and following the same procedure explained for $T > T_c$. This will give us

$$C(\vec{x}) = \begin{cases} \frac{T}{\pi\kappa} \ln|\vec{x}/a| & \text{if } x^2\mu^*/(\beta\kappa) \ll 1 \\ \frac{T}{m\pi\kappa} \ln|\vec{x}/a| & \text{if } x^2\mu^*/(\beta\kappa) \gg 1. \end{cases} \quad (5.12)$$

Using Eq. (4.28) we can write this as

$$C(\vec{x}) = \begin{cases} \frac{T}{\pi\kappa} \ln|\vec{x}/a| & \text{if } x \ll \xi^* \\ \frac{T_c}{\pi\kappa}\left(1 - e^{-4\pi\kappa q^*/T}\right)^{-1} \ln|\vec{x}/a| & \text{if } x \gg \xi^*, \end{cases} \quad (5.13)$$

where the correlation length ξ^* reads

$$\xi^* = \sqrt{\kappa/T\mu^*}. \quad (5.14)$$

Near T_c, $\xi^* \sim \exp(A/\epsilon)$, where A is a constant. In Eqs. (5.12)–(5.14), the constants κ and a have been restored.

A value for ξ is also obtained from the GV calculations. The static value is larger than the dynamic one for the same temperature [7,8] (in the continuous transition regime where they yield the same T_c). The difference arises from the fact that the two equations which determine q^* and m (the so-called "marginality condition") are replaced by two equations for the values of

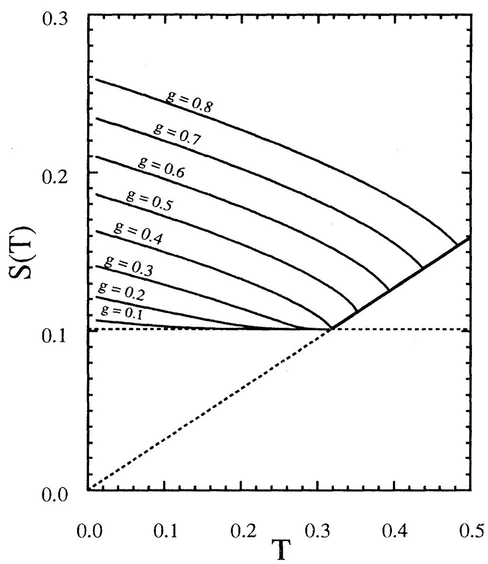

FIG. 8. The slope of the $\ln(x/a)$ term in Eq. (5.13) versus temperature for different coupling constants g, with $a = 1$ and $\kappa = 1$.

the two variational parameters within the one-step RSB. These yield a smaller "mass" and hence a larger correlation length.

For $g > g_{\rm tr}$, as T_c is approached from below, the slope $S(T)$ of the logarithm of the asymptotic behavior is given by

$$S(T) = \frac{1}{\pi^2}\left\{1 + \left(2\pi g^2/\kappa^2\right)^{1/\epsilon}\right\}, \qquad (5.15)$$

which is consistent with (but again, for the same reasons, different from) the general behavior found in GV calculations of Ref. [7].

Figure 8 shows the slope $S(T)$ for different values of g. For $g < g_{\rm tr}$ and near T_c, the behavior is also consistent with Ref. [8] and with the numerical results we obtained simulating the random-substrate model [14]. For $g > g_{\rm tr}$, the transition temperature is larger than $1/\pi$, and below T_c the slope has stronger dependence on g. In both cases, decreasing the temperature below T_c, the slope increases (see Fig. 8).

VI. CONCLUSIONS

In this paper we have studied the relaxational dynamics and equilibrium properties of the two-dimensional (2D) random-phase sine-Gordon model in the framework of the Martin-Siggia-Rose formalism. The analysis is based on the nonperturbative, self-consistent Hartree-type approximation of the nonlinear sine-Gordon term with the terms quadratic in the fields. The major results are related to the behavior of the autocorrelation function $q(t)$, response of the system $r(t)$, and the equilibrium height-height correlation function $C(\vec{x})$.

Analyzing the dynamic stability of the model we found that the fluctuation-dissipation theorem is violated below some critical temperature T_c for times larger than some characteristic time scale t^* which diverges with system size. Below T_c we separately studied dynamics for short times $t \ll t^*$ (where the FDT holds) and for large times $t \gg t^*$ where the dynamics becomes nonergodic, or the FDT solution becomes unstable. To stabilize the dynamics on all time scales, we followed an approach introduced by Horner [16] which is based on the generalization of the FDT to the so-called quasi-FDT, which assumes $m\partial_t q(t) = r(t)$, where a free parameter m is generally different on different time scales, and must be self-consistently determined by matching the proposed solutions for different times. For RSGM there is only one single time scale t^*. Using the QFDT we constructed the self-consistent, marginally stable (i.e., the system is always critical below T_c) solutions in the low-temperature phase.

The character of the transition between the ergodic ($T > T_c$) and the nonergodic ($T < T_c$) phase depends on the strength of the coupling constant g and it is reflected in the behavior of the $q(t)$. For $g < g_{\rm tr}$, as $T \to T_c$, $q(t)$ approaches its stationary value q_0 continuously (in this sense the transition is second order) while for $g > g_{\rm tr}$, $q(t)$ jumps discontinuously (a first-order transition) from q_0 to $q_c < q_0$. The value $g_{\rm tr}(= 1/\sqrt{2\pi})$, therefore, has a meaning of a tricritical point. In the regime of the second-order phase transition, the critical temperature is always $T_c = \kappa/\pi$, independent of g, while in the regime of the first-order phase transition it increases as g increases. In the latter case, the critical temperature is larger than the static one. Such a behavior was also found in other systems with a first-order transition [17,18] and it signals the existence of many metastable states [26].

Above T_c we rederived the well-known asymptotic results: logarithmic behavior of the autocorrelation function, $q(t) \sim T\ln(t)$, and the stationary height-height correlation function, $C(\vec{x}) \sim T\ln(x/a)$.

Below T_c, the self-consistent dynamics yields the following results. In equilibrium it essentially reproduces the results of the Gaussian variational [7,8] approach with the one-step replica-symmetry breaking. In the dynamics this is reflected by the breaking of the FDT for times above a single characteristic time t^*. Phase space is broken into separate components and the divergence of their barriers leads to the divergence of t^* in the thermodynamic limit. Within each component the autocorrelation $q(t)$ saturates to a finite value q^* algebraically: $q(t) = q^* - ct^{-\nu}$ as $t \to t^*$, where ν is a temperature-dependent exponent. q^* diverges and $\nu \to 0$ as $T \to T_c$.

The static height-height correlation function for large $|\vec{x}|$ behaves as $C(\vec{x}) \sim S(T)\ln(x/a)$. If $x \ll \xi^*$, the coefficient of the logarithmic term has the same temperature dependence as in the high-temperature phase: $S(T) \sim T$. For $x \gg \xi^*$, $S(T)$ has nontrivial T and g dependence. If $g < g_{\rm tr}$ and T close to T_c from below, $S(T) \approx 1/\pi^2$.

The results are obtained based on the self-consistent QFDT. The analysis applies if $t \to \infty$ before $L \to \infty$, where L is the system size. If the order of limits is reversed the system never reaches equilibrium below T_c. This will entail a nonstationary behavior with aging phenomena [27].

We summarize the analytic investigations of the RSGM dynamics. We observe the same "fault line" as mentioned in the Introduction concerning the equilibrium behavior. On one hand there are highly nontrivial dynamic RG results which indicate a *glassy ergodic* behavior below T_c. The dynamic is "critical" with a power-law relation between time and length scales $\tau(L) \sim L^{z(T)}$ and a nonlinear response $v \sim F^{z(T)/2}$ with a dynamic exponent $z(T) > 2$ which increases as T is lowered below T_c.

On the other hand, however, we have presented above a nonperturbative self-consistent approach which predicts the breaking of FDT and ergodicity below T_c (this approach is exact in the large-N limit [18]). The outlook for bridging between the two approaches is not very bright. A dynamic RG in which FDT is broken "by hand" is certainly possible and the results will undoubtedly follow their equilibrium counterparts [12]. A $1/N$ expansion is also possible but experience from other models makes it unlikely that the $1/N$ correction will change our results (and will certainly have a limited extrapolation to the $N = 1$ behavior).

Numerical simulations are probably the best hope but they are likely to clarify first the equilibrium situation. It should be emphasized, however, that the dynamic simulations with a small driving force F are in a much better agreement with the dynamic RG. Since the regime of smaller F is accessible (although it requires larger lattices and more disorder realizations), it may provide an alternative avenue to probe the potential FDT breaking as $F \to 0$. In that context an extension of the above calculations to the presence of $F \neq 0$ in order to derive the $v = v(F)$ relation will be very useful [28].

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